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ON THE "TEST PARTICLE" PROBLEM
FOR AN ELECTRON PLASMA

by

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Abstract

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Fokker-Planck coefficients for a test particle distribution in an equilibrium electron plasma are calculated from the generalized Balescu-Lenard equation. The results are found to disagree slightly from those of Rostoker and Rosenbluth.

AUTHOR

In this communication, we shall derive a Fokker-Planck equation for a "test particle" distribution from the many-species Balescu-Lenard equation. The friction and dispersion coefficients are found to differ somewhat from those previously calculated by Rostoker and Rosenbluth.[1] A reason for the discrepancies is advanced.

For a spatially uniform, many-species plasma, the Balescu-Lenard equation generalizes to[2]:

$$\frac{\partial f_i}{\partial t} = - \frac{\partial}{\partial \underline{v}_i} \cdot \underline{J}_i\,(\underline{v}_1) \tag{1}$$

where

$$\underline{J}_i\,(\underline{v}_1) = m_i \sum_j n_j \int d\underline{v}_2\, \underline{\underline{Q}}_{ij}\,(\underline{v}_1, \underline{v}_2) \cdot \left[\frac{f_j(\underline{v}_2)}{m_i} \frac{\partial f_i}{\partial \underline{v}_1} - \frac{f_i(\underline{v}_1)}{m_j} \frac{\partial f_j}{\partial \underline{v}_2} \right] \tag{2}$$

and

$$\underline{\underline{Q}}_{ij} = - \int \frac{d\underline{k}_1\, \underline{k}_1 \underline{k}_1}{m_i^2\, k_1^5} \frac{2(e_i e_j)^2\, \delta\left(\underline{k}_1 \cdot [\underline{v}_1 - \underline{v}_2] / k_1\right)}{\left| D^+\left(-\underline{k}_1, i\underline{k}_1 \cdot \underline{v}_1\right)\right|^2} \tag{3}$$

In these equations, m_i, e_i, n_i, and f_i are the mass, charge, number density, and velocity-space distribution function (normalized to unity) of the ith charge species. The plasma dispersion function D^+ is defined by

$$D^{+}(\underline{k},p) = 1 - \sum_{j} \frac{\omega_{pj}^{2}}{k^{2}} \int_{C} \frac{du}{u - ip/k} \frac{\partial F_{j}(u)}{\partial u} \tag{4}$$

where the contour C is from $-\infty$ to $-\infty$ for Re $p > 0$, but passes around the point $u = ip/k$ as p passes into its left half-plane. We also have the following definitions:

$$\omega_{pj}^{2} = \frac{4\pi n_j e_j^2}{m_j} \ , \quad F_j(u) = \int d\underline{v}\, f_j(\underline{v})\, \delta\left(u - \frac{\underline{k}\cdot\underline{v}}{k}\right) \ . \tag{5}$$

We now consider the case in which a very tenuous stream of "test" particles (charge Ze, mass M_t) is moving through an electron plasma, in equilibrium. There is assumed to be a uniform, immobile positive background; to allow discrete ions would only increase the algebra, and would not alter the technique.

For the electrons, we have the Maxwell distribution:

$$f_m(\underline{v}) = \left(\frac{m}{2\pi KT}\right)^{3/2} \exp\left(-\frac{mv^2}{2KT}\right) . \tag{6}$$

If the number density of the electrons is n_o and that of the test stream n_t, we express the smallness of the test particle concentration by

$$\frac{n_t}{n_o} \ll 1 \quad .$$

We shall assume that n_t is in fact so small that:

(i) interactions between test particles are negligible;

(ii) the test particles do not distort f_m, the distribution of "field" electrons.

To lowest significant order in n_t/n_o, it follows that (1) reduces to a Fokker-Planck equation (in the original sense):

$$\frac{\partial f_t(\underline{v}_1)}{\partial t} = -\frac{\partial}{\partial \underline{v}_1} \circ \left[\underline{F}(\underline{v}_1) f_t(\underline{v}_1) \right] + \frac{1}{2} \frac{\partial^2}{\partial \underline{v}_1 \partial \underline{v}_1} : \left[\underline{\underline{T}}(\underline{v}_1)\, f_t(\underline{v}_1) \right] \tag{7}$$

where the coefficients of dynamical friction and dispersion are given by

$$\underline{F}(\underline{v}_1) = -\frac{n_o M_t}{m} \int d\underline{v}' \, \underline{\underline{Q}} \, (\underline{v}_1, \underline{v}') \frac{\partial f_m(\underline{v}')}{\partial \underline{v}'} - n_o \frac{\partial}{\partial \underline{v}_1} \circ \int d\underline{v}' \, \underline{\underline{Q}} \, (\underline{v}_1, \underline{v}') \, f_m(\underline{v}') \tag{8}$$

$$\underline{\underline{T}}(\underline{v}_1) = -2n_o \int d\underline{v}' \, \underline{\underline{Q}} \, (\underline{v}_1, \underline{v}') \, f_m(\underline{v}') \; . \tag{9}$$

If we write

$$D^+ \left(-\underline{k}_1, \, i\underline{k}_1 \cdot \underline{v}_1 \right) \equiv 1 + \frac{\omega_{pe}^2}{k_1^2} \Psi \tag{10}$$

and note that Ψ depends only on the direction $\underline{k}_1/|\underline{k}_1|$, and not on $|\underline{k}_1|$, two of the integrals in (3) can be carried out. Thus

$$(\underline{\underline{Q}})_{\alpha\beta} = -\frac{2Z^2e^4}{M_t^2} \frac{1}{|\underline{v}_1 - \underline{v}'|} \oint d\phi \, \kappa_\alpha \kappa_\beta \int_0^{k_o} \frac{dk_1}{k_1} \frac{1}{\left|1 + \frac{\omega_{pe}^2}{k_1^2} \Psi \right|^2} \tag{11}$$

We have chosen the "one" coordinate axis along $\underline{v}_1 - \underline{v}'$, and the only non-vanishing components of $\underline{\underline{Q}}$ are Q_{22}, Q_{33}, and $Q_{32} = Q_{23}$ and κ_α is either the 2 or 3 component of a unit vector in the plane perpendicular to $\underline{v}_1 - \underline{v}'$. The integral in (11) has been cut off at k_o; $1/k_o$ is the distance of closest approach at which the integral is usually cut off[3,4], $1/k_o = e^2/KT$.

Now suppose we consider the case in which the test-particles have velocities considerably higher than the electron thermal speed $(KT/m)^{1/2}$. Then to the lowest significant order in $1/u_1$, where $u_1 = \underline{k}_1 \cdot \underline{v}_1 / k_1$,

$$\Psi \simeq -\frac{1}{u_1^2} - 2i\pi^2 \left(\frac{m}{2\pi KT}\right)^{3/2} u_1 \exp\left(-\frac{mu_1^2}{2KT}\right)$$

$$\equiv \Psi_r + i\Psi_i \tag{12}$$

say.

It is instructive to divide the wave number integration in (11) into two ranges, namely from k_D to k_o and 0 to k_D, where k_D is the Debye wave number, $(4\pi n_o e^2/KT)^{1/2}$. The result for $(\underline{\underline{Q}})_{\alpha\beta}$, the $\alpha\beta$ th component of $\underline{\underline{Q}}$, is:

$$(\underline{\underline{Q}})_{\alpha\beta} = -\frac{Z^2 e^4}{M_t^2} \frac{1}{|\underline{v}_1 - \underline{v}'|} \oint d\phi \, \kappa_\alpha \kappa_\beta (L+S) , \tag{13}$$

$$S = \ln\left|\frac{k_o^2 + \omega_{pe}^2 \psi}{k_D^2 + \omega_{pe}^2 \psi}\right| + \frac{\psi_r}{\psi_i} \tan^{-1}\left[\frac{\omega_{pe}^2 \psi_i}{(k_o^2 + \omega_{pe}^2 \psi_r)}\right] - \frac{\psi_r}{\psi_i} \tan^{-1}\left[\frac{\omega_{pe}^2 \psi_i}{(k^2 + \omega_{pe}^2 \psi_r)}\right] \tag{14}$$

(corresponding to $k_D \leqslant k_1 \leqslant k_o$), and

$$L = \ln\left|\frac{k_D^2 + \omega_{pe}^2 \psi}{\omega_{pe}^2 \psi}\right| + \frac{\psi_r}{\psi_i} \tan^{-1}\left[\frac{\omega_{pe}^2 \psi_i}{(k_D^2 + \omega_{pe}^2 \psi_r)}\right] - \frac{\psi_r}{\psi_i} \tan^{-1}\left[\frac{\psi_i}{\psi_r}\right] \tag{15}$$

from the range $0 \leq k_1 \leq k_D$.

Making use of (12), and that $|\psi_i| \ll |\psi_r|$ and $u_1 \gg (KT/m)^{1/2} \cong \omega_{pe} / k_D$,

$$S \cong \ln \left(\frac{k_o^2}{k_D^2} \right) \tag{16}$$

$$L \cong \ln \left(\frac{k_D^2\, u_1^2}{\omega_{pe}^2} \right) - 1 \quad . \tag{17}$$

If we neglect the slow ϕ dependence of the logarithms in L and S, the remaining integration in (13) can be done, and gives

$$\underline{\underline{Q}} = - \frac{Z^2 \pi e^4}{M_t^2} \left(\frac{g^2 \underline{\underline{I}} - \underline{g}\underline{g}}{g^3} \right) \left[\ln \left(\frac{k_o^2}{k_D^2} \right) + \left(\ln \left(\frac{k_D^2 v_1^2}{\omega_{pe}^2} \right) - 1 \right) \right] \tag{18}$$

where $\underline{g} = \underline{v}_1 - \underline{v}'$. With this expression for $\underline{\underline{Q}}$, the coefficient of friction readily simplifies to

$$\underline{F}(\underline{v}_1) = \frac{-Z^2 e^2 k_D^2}{M_t} \ln \left(\frac{k_o}{k_D} \right) \left(\frac{1}{m} + \frac{1}{M_t} \right) \frac{KT\, \underline{v}_1}{v_1^3}$$
$$+ \frac{Z^2 e^2 k_D^2}{2} \left(\frac{m+M_t}{M_t^2} \right) \frac{KT \underline{v}_1}{m v_1^3} \left(1 - \ln \left(\frac{m v_1^2}{KT} \right) \right) . \tag{19}$$

If we compare the expression (19) with equations (38) and (39) of reference 1 (the Rostoker-Rosenbluth expression for $F_{\parallel}$ should be $M_t \underline{F} \cdot \underline{v}_1 / v_1$ of our expression (19)), we find disagreement with the numerical coefficients.

We believe that these discrepancies could be due to some combination of the following: (i) there appear to have been terms dropped from equation (15) of reference 1 which are comparable with those retained, for this case; and (ii) more fundamentally, the multiple time scale aspects of the BBGKY formalism do not seem to have been consistently used. Contrary to the procedure of expanding the time derivative of the one-body distribution in powers of the plasma parameter, as is usually done[2,5], the one-body distribution itself has been expanded. Thus eq. (20) of reference 1 could not possibly lead to a Fokker-Planck equation in the original sense, insofar as the last two terms of it do not contain w_i at all, but only the zeroth order part of it. For purposes of this equation, these last two terms amount only to an inhomogeneous driving term.

In conclusion, we list for the sake of completeness the leading terms of $\underline{\underline{T}}$ also:

$$\underline{\underline{T}} = \frac{Z^2 e^2 k_D^2}{2M_t^2} \; KT \left(\frac{v_1^2 \underline{\underline{1}} - \underline{v}_1 \underline{v}_1}{v_1^3} \right) \left[\ln \left(\frac{k_o^2}{k_D^2} \right) + \left(\ln \left(\frac{k_D^2 v_1^2}{\omega_{pe}^2} \right) - 1 \right) \right] . \qquad (20)$$

FOOTNOTES

1. N. Rostoker and M. N. Rosenbluth, Phys. Fluids 3, 1 (1960).

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3. R. Balescu, Phys. Fluids 3, 52 (1960).

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5. N. N. Bogolyubov, Problems of a Dynamical Theory in Statistical Physics, State Technical Press, Moscow, 1946. English translation in Studies in Statistical Mechanics, Vol. 1, ed. by J. de Boer and G. E. Uhlenbeck, North-Holland, Amsterdam, 1962.